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On a theorem of S. Koshitani

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ON A THEOREM OF S. KOSHITANI

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Throughout G will represent a p-group of order p^r, and K a field of characteristic p. The nilpotency index of the radical $J(KG)$ of the group ring KG will be denoted by $t(G)$.

Recently, S. Koshitani [3, 4] proved that $t(G)$ takes the secondarily heighest value $p^{r-1}+p-1$ if and only if G contains an element of order p^{r-1}. His proof was completed by determining K-bases of $J(KG)^t$ $(t=1, 2, \cdots)$. One of the purposes of this paper is to present an alternative proof to the same by making use of Jennings' M-series (see [2, p. 182])

$$M_1 = G,\ \ M_t = \langle [M_{t-1},\ G],\ M^p_{(t/p)} \rangle$$

where (t/p) is the least integer which is not smaller than t/p. In fact, we can compute $t(G)$ by Jennings' formula

$$(1)\qquad t(G) = (\textstyle\sum_t td_t)(p-1)+1$$

where $p^{d_t} = |M_t/M_{t+1}|$ (see [2, Theorems 3. 7 and 5. 5]).

Furthermore, we shall present several results concerning $t(G)$.

1. Koshitani's theorem. If H is an abelian subgroup of G, c a central element of order p, and $\langle H,\ c\rangle$ is normal, then both $H^p\,(=\{h^p\,|\,h\in H\})$ and $\langle H^p,\ c\rangle$ are normal. The following theorem is fundamental in the subsequent study.

Theorem 1. *Assume that there exist an abelian subgroup H of exponent p^s and a central element c of order p such that $\langle H,\ c\rangle$ is normal and $G/\langle H^p,\ c\rangle$ is elementary. We set $p^b = [G:H]$ and $p^{c_t} = |H^{p^t}/H^{p^{t+1}}|$.*

(a) *If G/H^p is elementary, then*

$$(2)\qquad t(G) = (b+\textstyle\sum_{t=0}^{s-1} c_tp^t)(p-1)+1.$$

(b) *If G/H^p is not elementary but of exponent p, then*

$$(3)\qquad t(G) = (b+1+\textstyle\sum_{t=0}^{s-1} c_tp^t)(p-1)+1.$$

(c) *If the exponent of G/H^p is larger than p, then*

$$(4)\qquad t(G) = (b+p-1+\textstyle\sum_{t=0}^{s-1} c_tp^t)(p-1)+1.$$

Proof. Noting that $G' \subseteq \langle H^p, c \rangle$, one can easily see

(5) $$[\langle H^{p^t}, c \rangle, G] = [H^{p^t}, G] \subseteq H^{p^{t+1}} \quad (t \geqq 1).$$

(a) From the assumption, we see $G' \subseteq H^p$ and $G^p = H^p$. By the last and (5), we obtain the M-series;

$$\begin{aligned} M_2 = M_3 = \cdots &= M_p = H^p \\ M_{p+1} = \cdots\cdots &= M_{p^2} = H^{p^2} \\ &\cdots\cdots \\ M_{p^{s-2}+1} = \cdots\cdots &= M_{p^{s-1}} = H^{p^{s-1}} \\ M_{p^{s-1}+1} &= 1. \end{aligned}$$

Hence, (2) follows from (1).

(b) Since $G/\langle H^p, c \rangle$ is elementary, we obtain $H^p \subseteq \langle G', H^p \rangle \subseteq \langle H^p, c \rangle$. Now, recalling that G/H^p is not elementary but of exponent p, one will easily see that $\langle G', H^p \rangle = \langle H^p, c \rangle$ and $G^p = H^p$. Hence, by (5) and the last, we obtain the M-series;

In case $p \neq 2$:
$$\begin{aligned} M_2 &= \langle H^p, c \rangle \\ M_3 = \cdots\cdots &= M_p = H^p \\ M_{p+1} = \cdots\cdots &= M_{p^2} = H^{p^2} \\ &\cdots\cdots \\ M_{p^{s-2}+1} = \cdots &= M_{p^{s-1}} = H^{p^{s-1}} \\ M_{p^{s-1}+1} &= 1. \end{aligned}$$

In case $p = 2$:
$$\begin{aligned} M_2 &= \langle H^2, c \rangle \\ M_3 = M_4 &= H^4 \\ M_5 = \cdots\cdots &= M_8 = H^8 \\ &\cdots\cdots \\ M_{2^{s-2}+1} = \cdots &= M_{2^{s-1}} = H^{2^{s-1}} \\ M_{2^{s-1}+1} &= 1. \end{aligned}$$

Since $|G/\langle H^p, c \rangle| = p^{b+e_0-1}$ by
$$p^{b+e_0} = |G/H^p| = |G/\langle H^p, c \rangle| \cdot |\langle H^p, c \rangle H^p|,$$
in either cases (3) follows from (1).

(c) By $G' \subseteq \langle H^p, c \rangle = \langle G^p \rangle$ and (5), we obtain the M-series;
$$\begin{aligned} M_2 = \cdots\cdots &= M_p = \langle H^p, c \rangle \\ M_{p+1} = \cdots\cdots &= M_{p^2} = H^{p^2} \end{aligned}$$

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$$M_{p^{s-2}+1} = \cdots = M_{p^{s-1}} = H^{p^{s-1}}$$
$$M_{p^{s-1}+1} = 1.$$

Thus, (4) follows from (1).

In Theorem 1, if $H = \langle h \rangle$ is a cyclic group of order p^s then $c_t = 1$ for all t, and so $(\sum_t c_t p^t)(p-1) = p^s - 1$. Hence, we readily obtain the next corollary which will play an important role in our alternative proof of Koshitani's theorem and whose first assertion generalizes a result of R. Holvoet [1, Stelling 3] (see also the proof of 3) $\Longrightarrow$ 1) in Theorem 4).

Corollary 2. *Assume that there exist a central element $c \in G$ of order p and an element $h \in G$ of order p^s such that $\langle h, c \rangle$ is normal and $G/\langle h^p, c \rangle$ is elementary.*

(a) *If $G/\langle h^p \rangle$ is elementary, then $t(G) = p^s + (r-s)(p-1)$.*

(b) *If $G/\langle h^p \rangle$ is not elementary but of exponent p, then $t(G) = p^s + (r-s+1)(p-1)$.*

(c) *If the exponent of $G/\langle h^p \rangle$ is larger than p, then $t(G) = p^s + (r-s+p-1)(p-1)$.*

Although the next lemma is familiar more or less, for the sake of completeness, we shall give here the proof.

Lemma 3. *Let $q \geq 2$ and $n \geq 2$ be positive integers, and $r = \sum_{i=1}^n e_i$ with positive integers $e_1 \geq e_2 \geq \cdots \geq e_n$. If $\sum_{i=1}^n q^{e_i} - (n-1) \geq q^{r-1}$, then* (1) $n = 3$, $r = 3$, $q = 2$ *or* (2) $n = 2$, $e_2 = 1$ (*and conversely*).

Proof. Obviously, if $x \geq 2$ and $y \geq t \geq 2$ then $xy \geq x+y+t-2 \geq x+y$. Assume $n \geq 3$. Then by the above remark

$$\begin{aligned}
0 &\geq q^{r-1} - \textstyle\sum_{i=1}^n q^{e_i} + n - 1 \\
&\geq q^{e_1+e_2-1} + q^{e_3+\cdots+e_n} + q^{n-2} - 2 - \textstyle\sum_{i=1}^n q^{e_i} + n - 1 \\
&\geq q^{e_1+e_2-1} - q^{e_1} - q^{e_2} + (n-2)q + n - 3 \\
&= (q^{e_1} - q)(q^{e_2-1} - 1) + (n-3)(q+1) \geq 0.
\end{aligned}$$

Thus, $n=3$, $e_2 = e_3 = 1$ and $(q^{r-2}-2)(q-1) = 0$, whence it follows $q=2$, $r=3$. Finally, if $n=2$ then $q - 1 \geq (q^{e_1} - q)(q^{e_2-1} - 1)$, and so $e_2 = 1$.

Now, we shall prove Koshitani's theorem.

Theorem 4. *Assume that G is not cyclic. Then the following are equivalent:*

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1) $t(G) = p^{r-1} + p - 1$.
2) $t(G) > p^{r-1}$.
3) *G contains an element of order* p^{r-1}.

Proof. $1) \Longrightarrow 2)$: Trivial.

$3) \Longrightarrow 1)$: Let h be an element of order p^{r-1}. Then $\langle h \rangle$ is normal. If $G/\langle h^p \rangle$ is cyclic, then there exists an element x such that $x^p = h$, and so G is cyclic. Thus, $G/\langle h^p \rangle$ is elementary and $t(G) = p^{r-1} + p - 1$ by Corollary 2 (a).

$2) \Longrightarrow 3)$: Assume that G is an abelian group of type $(p^{e_1}, p^{e_2}, \cdots, p^{e_n})$ $(e_1 \geq e_2 \geq \cdots \geq e_n)$. Then, by [5, Theorem], $t(G) = \sum_{k=1}^{n} p^{e_k} - (n-1) > p^{r-1}$, where $r = \sum_{k=1}^{n} e_k$, and $n \geq 2$. By Lemma 3, this inequality holds only for $n = 2$, $e_1 = r-1$, $e_2 = 1$. Thus, G is of type (p^{r-1}, p), and so G contains an element of order p^{r-1}. Henceforth, we assume that G is non-abelian $(r \geq 3)$. We shall proceed by the induction on r. Every non-abelian group of order p^3 contains an element of order p^2, provided $G \neq M(p) = \langle a, b, c \mid a^p = b^p = c^p = 1, a^{-1}ba = bc, a^{-1}ca = c, b^{-1}cb = c \rangle$ $(p \geq 3)$. However, $t(M(p)) = 4p - 3 \leq p^2$ by Corollary 2 (b). Thus, for $r = 3$, $2) \Longrightarrow 3)$. Now, assume $r \geq 4$. Let c be a central element of order p, and $\overline{G} = G/\langle c \rangle$. Then $\overline{G}$ is not cyclic and $t(\overline{G}) > p^{r-2}$ (see [6, Theorem 2. 4]). Hence, by the induction hypothesis, there exists an element $\overline{h}$ of order p^{r-2} in $\overline{G}$. Since $\overline{G}$ is not cyclic, $\overline{G}/\langle \overline{h}^p \rangle$ is elementary (see $3) \Longrightarrow 1)$). Hence, there exists an element h such that $\langle h, c \rangle$ is normal and $G/\langle h^p, c \rangle$ is elementary. If $h^{p^{r-2}} = c^f$ $(0 < f < p)$ then h is of order p^{r-1}. While, if h is of order p^{r-2} then, by Corollary 2, $t(G) = p^{r-2} + 2(p-1)$ or $p^{r-2} + 3(p-1)$ or $p^{r-2} + p^2 - 1$, and hence $t(G) > p^{r-1}$. This contradiction completes our induction.

2. Further results. First, we shall characterize G with $t(G) = p^{r-1}$.

Theorem 5. *If* $t(G) = p^{r-1}$ *then* G *is* $M(3)$ *or an elementary abelian group of order* 8 (*and conversely*).

Proof. Case I. G is abelian. Let G be of type $(p^{e_1}, p^{e_2}, \cdots, p^{e_n})$ $(e_1 \geq e_2 \geq \cdots \geq e_n)$. Then $n \geq 2$, and $p^{r-1} = t(G) = \sum_{i=1}^{n} p^{e_i} - (n-1)$ by [5, Theorem]. Hence, by Lemma 3, G is of type (2, 2, 2).

Case II. G is non-abelian and $r = 3$. Since G is of exponent p by Theorem 4, it follows $G = M(p)$. However, $p^2 = t(G) = 4p - 3$ by Corollary 2 (b). Thus $p = 3$.

Case III. G is non-abelian and $r \geq 4$. Let G be an example of minimal order with $t(G)=p^{r-1}$ in this case, $x\in G$ a central element of order p, and $\bar{G}=G/\langle x\rangle$. Then $t(\bar{G})\geq p^{r-2}$ by [6, Theorem 2.4]. If $t(\bar{G})>p^{r-2}$, then $\bar{G}$ contains an element $\bar{h}$ of order p^{r-2} (Theorem 4), and so there exists an element $h\in G$ of order p^{r-1} or p^{r-2} such that $G/\langle x, h^p\rangle$ is elementary. Thus, by Corollary 2, $p^{r-1}=t(G)=p^{r-1}+p-1$ or $p^{r-2}+2(p-1)$ or $p^{r-2}+3(p-1)$ or $p^{r-2}+p^2-1$. This contradiction means $t(\bar{G})=p^{r-2}$. Next, we shall show that $r=4$. In fact, if $r\geq 5$ then $\bar{G}$ is abelian and $|\bar{G}|=2^3$ (Case I), which contradicts $r\geq 5$. Hence $|\bar{G}|=p^3$ and $t(\bar{G})=p^2$, so that $\bar{G}=M(3)$ or an elementary abelian group of order 8 (Case II). However, if G is the latter then $t(G)=5$ or 6 by Corollary 2 (see also Theorem 6). This contradicts $t(G)=2^3$. While, if $\bar{G}=\langle a, \bar{b}, \bar{c}\,|\,\bar{a}^3=\bar{b}^3=\bar{c}^3=1,\ a^{-1}\bar{b}\bar{a}=\bar{b}\bar{c},\ \bar{a}^{-1}\bar{c}\bar{a}=\bar{c},\ \bar{b}^{-1}\bar{c}\bar{b}=\bar{c}\rangle$, then $[G,G]\subseteq\langle x, c\rangle$, $G^3\subseteq\langle x\rangle$, $[\langle x, c\rangle, G]\subseteq\langle x\rangle$. Hence we see that

$$1\neq M_2\subseteq\langle x, c\rangle,\ M_3\subseteq\langle x\rangle,\ M_4=M_5=M_6=\langle M_2^3\rangle,\ M_7=1,$$

whence it follows $2\leq d_1\leq 3$, $d_2\leq 2$, $d_3\leq 1$, $d_4=d_5=0$ and $d_6\leq 1$. But this yields a contradiction $3^3=t(G)=(d_1+2d_2+3d_3+6d_6)(3-1)+1<3^3$.

Next, we shall show that the secondarily lowest value of $t(G)$ is $(r+1)(p-1)+1$, which is given in the next

Theorem 6. *If G is not elementary then the following are equivalent:*
1) $t(G)=(r+1)(p-1)+1$.
2) $t(G)<(r+2)(p-1)+1$.
3) *There exists a central element c of order p such that $G/\langle c\rangle$ is elementary. When this is the case, $p=2$ or the exponent of G is p.*

Proof. 1) $\Longrightarrow$ 2) is trivial and 3) $\Longrightarrow$ 1) is clear by Corollary 2 (b), (c).
2) $\Longrightarrow$ 3): By formula (1), the condition 2) implies $2>\sum_t (t-1)d_t=d_2+2d_3+\cdots+(t-1)d_t$ for some t and hence $d_3=\cdots=d_t=0$, $d_2=0$ or 1. $d_2=0$ means that $M_2=1$ and so G is elementary. Thus, $d_2=1$ and $d_1=r-1$, which means $t(G)=(r+1)(p-1)+1$. Moreover, M_2 is a central subgroup of order p. Let c be a generator of M_2. Then $G/\langle c\rangle$ is elementary. If the exponent of G is greater than p, by Corollary 2 (c), $(r+p-1)(p-1)+1=t(G)=(r+1)(p-1)+1$ and so $p=2$.

S. Koshitani computed also $t(G)$ for G meta-cyclic. Here, we shall do the same by making use of M-series.

Theorem 7. *Let G be a meta-cyclic group with generators a, b such* *that* $\langle b \rangle$ *is a normal subgroup of order* p^n *and* $a^{p^{r-n}} = b^{p^k}$ $(0 \leq k \leq n)$. *Then,*

$$t(G) = \begin{cases} p^n + p^{r-n} - 1 & \text{if } r - n \leq k \\ p^{r-k} + p^k - 1 & \text{if } r - n \geq k. \end{cases}$$

Proof. If G is cyclic, then the assertion is trivial. Henceforth, we assume that $m = r - n \geq 1$ and $k \geq 1$. Noting that $[a, b]$ is contained in $\langle b^p \rangle$, one can easily see that $\langle \langle a^{p^t}, b^{p^t} \rangle^p \rangle = \langle a^{p^{t+1}}, b^{p^{t+1}} \rangle$ and $[a^{p^t}, G] \leq [\langle a^{p^t}, b^{p^t} \rangle, G] \leq \langle b^{p^{t+1}} \rangle$. Hence, we obtain the M-series of G;

In case $m \leq k$:

$$\begin{aligned}
M_2 &= \cdots\cdots = M_p = \langle a^p, b^p \rangle \\
M_{p+1} &= \cdots\cdots = M_{p^2} = \langle a^{p^2}, b^{p^2} \rangle \\
&\cdots\cdots \\
M_{p^{m-2}+1} &= \cdots = M_{p^{m-1}} = \langle a_{p^{m-1}}, b_{p^{m-1}} \rangle \\
M_{p^{m-1}+1} &= \cdots\cdots = M_{p^m} = \langle b_{p^m} \rangle \\
&\cdots\cdots \\
M_{p^{n-2}+1} &= \cdots\cdots = M_{p^{n-1}} = \langle b_{p^{n-1}} \rangle \\
M_{p^{n-1}+1} &= 1.
\end{aligned}$$

In case $m \geq k$:

$$\begin{aligned}
M_2 &= \cdots\cdots = M_p = \langle a^p, b^p \rangle \\
M_{p+1} &= \cdots\cdots = M_{p^2} = \langle a^{p^2}, b^{p^2} \rangle \\
&\cdots\cdots \\
M_{p^{k-2}+1} &= \cdots\cdots = M_{p^{k-1}} = \langle a^{p^{k-1}}, b^{p^{k-1}} \rangle \\
M_{p^{k-1}+1} &= \cdots = M_{p^k} = \langle a^{p^{k-1}} \rangle \\
&\cdots\cdots \\
M_{p^{r-k-2}+1} &= \cdots\cdots = M_{p^{r-k-1}} = \langle a^{p^{r-k-1}} \rangle \\
M_{p^{r-k-1}+1} &= 1.
\end{aligned}$$

Now, from (1) one will easily obtain $t(G)$.

In [4] S. Koshitani has characterize G with $t(G) = 4, 5, 6$. The next gives a characterization of G with $t(G) = 7$.

Proposition 8. $t(G) = 7$ *if and only if* G *is one of the following groups*:

1) *A cyclic group of order* 7.
2) *An elementary abelian group of order* 3^3 *or* 2^6.
3) *An abelian group of type* $(2^2, 2^2)$

4) *A non-elemeniary group of order* 2^5 *which contains a central involution* c *such that the factor group modulo* $\langle c \rangle$ is elementary.

5) $\langle a,\ b \mid a^4 = b^4 = 1,\ a^{-1}ba = b^3 \rangle$.

6) $\langle a,\ b,\ c \mid a^2 = b^2 = c^4 = 1,\ a^{-1}ba = b,\ a^{-1}ca = bc,\ b^{-1}cb = c \rangle$.

Proof. By (1), $6 \geq r(p-1)$ and $p-1$ is a divisor of 6. Thus the order of G is 7 or 3^r $(r \leq 3)$ or 2^r $(r \leq 6)$. The result follows from Corollary 2, Theorems 4, 6 and the table of defining relations of groups of order 2^4 (see e.g. W. Burnside's text, Theory of Groups of Finite Order).

Remark. If $t(G)$ is even then $p=2$, for $p-1$ is a divisor of $t(G)-1$ by (1). Hence, Koshitani's characterization of G with $t(G)=4$, 6 is easy by Corollary 2. If $t(G)=2^n+1$ then $p=2^{2^k}+1$ (Fermat number).

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